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Yukawa Couplings Between (2,1)-Forms

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Abstract

The compactification of superstrings leads to an effective field theory for which the space-time manifold is the product of a four dimensional Minkowski space with a six dimensional Calabi-Yau space. The particles that are massless in the four dimensional world correspond to differential forms of type (1,1) and of type (2,1) on the Calabi-Yau space. The Yukawa couplings between the families correspond to certain integrals involving three differential forms. For an important class of Calabi-Yau manifolds, which includes the cases for which the manifold may be realized as a complete intersection of polynomial equations in a projective space, the families correspond to (2,1)-forms. The relation between (2,1)-forms and the geometrical deformations of the Calabi-Yau space is explained and it is shown, for those cases for which the manifold may be realized as the complete intersection of polynomial equations in a single projective space or for many cases when the manifold may be realized as the transverse intersection of polynomial equations in a product of projective spaces, that the calculation of the Yukawa coupling reduces to a purely algebraic problem involving the defining polynomials. The generalization of this process is presented for a general Calabi-Yau manifold.

1. Introduction

In the low energy regime superstring theory is believed to reduce to an effective ten-dimensional field theory propagating on a spacetime manifold which is the product of a four dimensional Minkowski space with a Calabi-Yau space, i.e., a compact Kähler manifold of vanishing first Chern class and complex dimension three. The massless families and antifamilies correspond to harmonic (1,1) and (2,1) forms of the internal manifold [1]. The Yukawa couplings arise as integrals of triples of harmonic forms[2]. The precise form of the integral depends on whether the three zero modes correspond to (1,1)-forms or (2,1)-forms. If they correspond to three (1,1) forms a,b,c, say, then the coupling is

$$\kappa = \int_M a \wedge b \wedge c. \tag{1.1}$$

As has been pointed out by Strominger this leads, in this case, to an interesting topological interpretation for the coupling. Through deRham cohomology each of the two-forms is associated with a four-surface. In a six (real) dimensional space three four-surfaces meet in some number of points. There exists a basis in which $\kappa(a,b,c)$ is the number of points of intersection of these surfaces. Couplings of this type have been discussed for a number of cases by Strominger [3] and it is not our purpose to dwell further on them here, rather this article is devoted to a discussion of the couplings for the case that the zero modes are related to (2,1)-forms which is the case for many manifolds of interest.

One might reasonably enquire how it is that three 3-forms can be integrated over a six-dimensional manifold. The way in which this comes about depends on the special properties of Calabi-Yau spaces. It is a crucial property of these spaces that there exists a nowhere zero holomorphic 3-form. That is a nowhere zero 3-form

$$\Omega = \frac{1}{3!}\,\Omega_{\mu\nu\rho}(x)dx^{\mu}\wedge dx^{\nu}\wedge dx^{\rho} \qquad (1.2)$$

which has $\Omega_{\mu\nu\rho}$ as its only nonzero components[f1] and where the $\Omega_{\mu\nu\rho}$ are moreover holomorphic functions of the coordinates.

Given Ω it is possible to define for every (2,1)-form

$$\omega = \frac{1}{2}\,\omega_{\mu\nu\bar{\rho}}\,dx^{\mu}\wedge dx^{\nu}\wedge dx^{\bar{\rho}} \qquad (1.3)$$

an equivalent object

$$\omega^{\mu} = \frac{1}{2}\,\bar{\Omega}^{\mu\rho\sigma}\omega_{\rho\sigma\bar{\nu}}\,dx^{\bar{\nu}}. \qquad (1.4)$$

ω^{μ} is a (0,1)-form with a holomorphic vector index. It may be regarded as a (0,1)-form that takes values in the holomorphic tangent bundle T. It is straightforward to show that the original (2,1)-form $\omega_{\rho\sigma\bar{\nu}}$ is harmonic with respect to the exterior derivative d if an only if ω^{μ} is harmonic with respect to $\bar{\partial}$, the antiholomorphic part of d. The Yukawa coupling involves three such forms a^{μ}, b^{ν}, c^{ρ}, say, and takes the explicit form [2]

$$\kappa = \int\Omega\wedge a^{\mu}\wedge b^{\nu}\wedge c^{\rho}\Omega_{\mu\nu\rho} \,. \qquad (1.5)$$

The purpose of this article is to discuss the significance of the coupling in relation to deformations of the complex structure of M and to evaluate it for

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f1. Notation: We shall use x^{μ}, $\mu = 1,2,3$ to denote the coordinates of M, Latin indices m,n.. will refer to a real coordinate basis and run over six values

a number of cases of interest. We shall see that for the case that M may be realized as a complete intersection of polynomial equations in a projective space, and in a number of other cases, the evaluation of κ reduces to a purely algebraic problem.

Much of what follows is material that is already present in the literature, though some is new. The discussion of the deformation of complex structure draws heavily on the classic work of Kodaira and Spencer [4,5]. The relationship between the holomorphic three-form and the defining polynomials was known to the mathematicians but I learnt it from the articles by Witten [6] and Strominger and Witten [7]. I review these relations here in order to make the account reasonably self contained. Witten [6] also showed that the Yukawa couplings are strongly constrained by the pseudosymmetries of the manifold. Finally the fact that statements about the cohomology classes of algebraic varieties translate into algebraic statements is a fact I learnt from conversations with Bott.

The process of evaluating κ breaks naturally into two parts. The first is to understand the geometrical meaning of the integral and the second is its explicit evaluation.

2. The geometrical meaning of $H^{(0,1)}_{\bar\partial}(M,T)$

Let us suppose initially that M is given as a hypersurface in an (N+3) dimensional projective space P_{N+3} defined as the transverse intersection of N polynomials

$$p^{\alpha}(z) = 0 \quad , \quad \alpha = 1,\cdots,N \tag{2.1}$$

where z^A, $A = 1,\cdots,N+4$ are homogeneous coordinates for the projective space and the degrees of the polynomials p^α are so chosen that the hypersurface has vanishing first Chern class. If the hypersurface has vanishing first Chern class then, in virtue of Yau's theorem, we may take it to be endowed with a Ricci-flat metric g_{mn}

$$R_{mn}(g) = 0 \tag{2.2}$$

Consider an infinitesimal variation of M which takes us to another nearby Ricci-flat manifold with metric $g_{mn} + h_{mn}$

$$R_{mn}(g+h) = 0 \tag{2.3}$$

Linearization of this equation and imposition of the coordinate condition $\nabla^m h_{mn} = 0$ leads to a differential equation for h_{mn}, the Lichnerowicz equation,

$$\Delta_L h_{mn} \equiv h_{mn} + 2R_m{}^r{}_n{}^s h_{rs} = 0 . \tag{2.4}$$

The first point to note is that, owing to the special proprties of Kähler spaces the Lichnerowicz operator Δ_L does not mix modes of different type i.e., the $h_{\mu\bar{\nu}}$, $h_{\mu\nu}$, $h_{\bar{\mu}\bar{\nu}}$ parts of h_{mn} separately satisfy the equation.